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Linear widths of a multivariate function space equipped with a Gaussian measure

Chen Guanggui[a, b], Fang Gensun[a, *]

[a] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
[b] School of Computers and Mathematical-Physical Science, XiHua University, Chengdu, China

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Abstract

We determine the asymptotic values on the linear probabilistic (N, δ)-widths and linear p-average N-widths of the space of multivariate functions with bounded mixed derivative $MW_2^r(\mathbb{T}^d)$, $r = (r_1, \ldots, r_d)$, $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$, equipped with a Gaussian measure μ in $L_q(\mathbb{T}^d)$. That is, the following asymptotic equivalences hold:

(1) If $1 < q \leqslant 2$, then

$$\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \asymp (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\left(\ln^{(\nu-1)/2} N\right) \times \sqrt{1+(1/N)\ln(1/\delta)}.$$

(2) If $1 < q < \infty$, then

$$\lambda_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \asymp (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1/2)}\left(\ln^{(\nu-1)/2} N\right).$$

Here $0 < \delta \leqslant 1/2$, and $\rho > 1$ depends only on the eigenvalues of the correlation operator of the measure μ (see (4)).

* Corresponding author.
E-mail address: fanggs@bnu.edu.cn (F. Gensun).

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If the dimension $d \geqslant 2$, then the asymptotic exact order of probabilistic linear widths of $MW_2^r(\mathbb{T}^d)$ with the Gaussian measure μ in the $L_q(\mathbb{T}^d)$ space for the cases $q = 1$, $2 < q \leqslant \infty$; and the average linear widths $\lambda_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)$ for the cases $q = 1$ and $q = \infty$ are still open.

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1. Introduction and main results

Let W be a bounded subset of a normed linear space X with norm $\|\cdot\|$, and F_N be a N-dimensional subspace of X. If W is a bounded set of X, the quantity

$$e(W, F_N, X) := \sup_{x \in W} e(x, F_N, X),$$

where

$$e(x, F_N, X) := \inf_{y \in F_N} e(x, y, X) := \inf_{y \in F_N} \|x - y\|$$

is called the deviation of W from F_N. It shows how well the "worst" elements of W can be approximated by F_N, however, another choice of X_n might provide a smaller deviation. Thus, we shall consider the possibility of allowing the N-dimensional subspaces to very within X. This idea, introduced by Kolmogorov [10], is now referred to as the N-width, in the sense of Kolmogorov, or as the N-width of W in X, which is given by

$$d_N(W, X) = \inf_{F_N} e(W, F_N, X) = \inf_{F_N} \sup_{x \in W} \inf_{y \in F_N} \|x - y\|, \tag{1}$$

where F_N runs through all possible linear subspaces of X of dimension at most N.

Let T be a linear operator from X to X, and denote by

$$\lambda(W, T, X) = \sup_{x \in W} \|x - Tx\|$$

as the linear distance of image TW from the set W. The linear N-width of the set W in X is defined by

$$\lambda_N(W, X) = \inf_{T_N} \lambda(W, T_N, X),$$

where T_N runs over all linear operator from X to X with rank at most N.

Assume that W contains a Borel field $\mathcal{B}$ consisting of open subsets of W and is equipped with a probability measure μ defined on $\mathcal{B}$. That is, μ is a σ-additive nonnegative function on $\mathcal{B}$, and $\mu(W) = 1$. Let $\delta \in [0, 1)$ be an arbitrary number. The corresponding probabilistic

Kolmogorov (N, δ)-width and probabilistic linear (N, δ)-width of a set W with a measure μ in the space X are, respectively, defined, by

$$
\begin{aligned}
d_{N,\delta}(W, \mu, X) &= \inf_{G_\delta} d_N(W \backslash G_\delta, X), \\
\lambda_{N,\delta}(W, \mu, X) &= \inf_{G_\delta} \lambda_N(W \backslash G_\delta, X),
\end{aligned}
\tag{2}
$$

where G_δ runs through all possible subsets in $\mathcal{B}$ with measure $\mu(G_\delta) \leqslant \delta$. The p-average Kolmogorov N-width and p-average linear N-width are defined by

$$
\begin{aligned}
d_N^{(a)}(W, \mu, X)_p &= \inf_{F_N} \left(\int_W e(x, F_N)^p \, d\mu(x) \right)^{1/p}, \qquad 0 < p < \infty, \\
\lambda_n^{(a)}(W, \mu, X)_p &= \inf_{T_n} \left(\int_W \|x - T_n x\|_X^p \, d\mu(x) \right)^{1/p}, \quad 0 < p < \infty.
\end{aligned}
\tag{3}
$$

The classical N-width of the class of functions characterizes the optimal error of the hardest elements in the worst case setting. In the probabilistic approach, the error is defined by the worst case performance on a subset of measure at least $1 - \delta$, so the probabilistic width can be understood as the μ-distribution of the approximation on all subsets of W which reflects the intrinsic structure of the class. Therefore, probabilistic case setting, compared with the worst case setting, allows one to give deeper analysis of the smoothness and approximation for the function class.

We see in the average case approach that the error is defined by the integral with respect to a given probability measure μ. Here the approximation emphasizes not the elements which attain the supremum and may be very small in measure, but the elements on which the given measure is most concentrated. So the average width characterizes the optimal approximation of the "most" elements of classes by N-dimensional subspaces.

Detailed information about the usual widths, such as the Kolmogorov widths, Gel'fand widths and linear widths, may be found in the books [22,26]. Quantities similar to (3) were considered in [27]. The study of probabilistic and average widths has been suggested only recently (see e.g., [18,19,27,28]) and relatively few results have been obtained. Moreover, the majority of the results obtained so far are for univariate classes of functions ($d = 1$) (see, e.g., [3,4,7,14–17,23]). These include results on probabilistic and average Kolmogorov and linear widths of one dimensional Sobolev classes of functions in the L_q-norm, $1 \leqslant q \leqslant \infty$. Papageorgiou and Wasilkowski [20], Woźniakowski [29], Paskov [21], Hickernell and Woźniakowski [8], Kühn and Linde [11] have investigated the problems of integration and approximation of functions that depend on d variables. In the monographs of Traub et al. [27], Ritter [24], some other different problems, which have closely related with probabilistic width and average width, such as ε-complexity and the minimal error of the problems of function approximation and integration by using N standard information, and the problem of approximation solution of integral and differential equations, were investigated in the worst, average, probabilistic case setting, and randomized setting. Furthermore, Ritter [24] provided the most recent and very detailed survey of average case setting results.

Denote by $L_q(\mathbb{T}^d)$, $1 \leqslant q \leqslant \infty$, the classical q-integral Lebesgue space of 2π-periodic functions with the usual norm $\|\cdot\|_{L_q^d} := \|\cdot\|_{L_q(\mathbb{T}^d)}$. Let $y = (y_1, \ldots, y_d)$, $t = (t_1, \ldots, t_d) \in$

$\mathbb{R}^d$, $s \in \mathbb{R}$. Then we write $|y|^s = |y_1|^s \dots |y_d|^s$, $y + s = (y_1 + s, \dots, y_d + s)$, $y > s$ which means that $y_j > s$, $j = 1, \dots, d$, and $(y, t) = \sum_{j=1}^d y_j t_j$.

Consider the Hilbert space $L_2(\mathbb{T}^d)$ consisting of all 2π-periodic functions x defined on the d-dimensional torus $\mathbb{T}^d := [0, 2\pi)^d$ ($\mathbb{T} := \mathbb{T}^1$) with the Fourier series

$$x(t) = \sum_{k \in \mathbb{Z}^d} c_k \exp(i(k, t)) = \sum_{k \in \mathbb{Z}^d} c_k e_k(t), \quad e_k(t) := \exp(i(k, t))$$

and inner product

$$\langle x, y \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} x(t) \overline{y(t)} \, dt, \qquad (x, y \in L_2(\mathbb{T}^d)).$$

For arbitrary vector $r = (r_1, \dots, r_d) \in \mathbb{R}^d$, we define the rth-order derivative of x in the sense of Weyl by

$$x^{(r)}(t) := (D^r x)(t) = \sum_{k \in \mathbb{Z}^d} (ik)^r c_k \exp(i(k, t)),$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, and $(ik)^r = \prod_{j=1}^d |k_j|^{r_j} \exp((\pi i/2) \operatorname{sgn} r_j)$.

We study the multivariate Sobolev space with mixed derivative $MW_2^r(\mathbb{T}^d)$, $r = (r_1, \dots, r_d) \in \mathbb{R}_+^d$, which consists of all functions $x \in L_2(\mathbb{T}^d)$ satisfying the additional condition

$$\int_0^{2\pi} x(t) \, dt_j = 0, \qquad j = 1, \dots, d,$$

which means that $c_k = 0$, if $k = (k_1, \dots, k_d)$, and $k_1 k_2 \dots k_d = 0$. The space $MW_2^r(\mathbb{T}^d)$ is a Hilbert space with the inner product

$$\langle x, y \rangle_r := \langle x^{(r)}, y^{(r)} \rangle$$

and the norm $\|x\|^2_{MW_2^r(\mathbb{T}^d)} = \langle x^{(r)}, x^{(r)} \rangle$. It is well known that if $r > \max\{0, 1/2 - 1/q\}$, then the space $MW_2^r(\mathbb{T}^d)$ can be embedded continuously into the $L_q(\mathbb{T}^d)$, $1 \leqslant q \leqslant \infty$.

We equip $MW_2^r(\mathbb{T}^d)$ with a Gaussian measure μ whose mean is zero and whose correlation operator C_μ has eigenfunctions $e_k = \exp(i(k, \cdot))$ and eigenvalues

$$\lambda_k = |k|^{-\rho}, \qquad \rho > 1, \tag{4}$$

that is

$$C_\mu e_k = \lambda_k e_k, \qquad \forall k \in \mathbb{Z}_0^d,$$

where

$$\mathbb{Z}_0^d = \left\{ k = (k_1, \dots, k_d) \in \mathbb{Z}^d : k_i \neq 0, \ i = 1, \dots, d \right\}.$$

Let $y_1, \dots, y_n$ be any orthogonal system of functions in $L_2(\mathbb{T}^d)$, $\sigma_j = \langle C_\mu y_j, y_j \rangle$, $j = 1, \dots, n$, and B be an arbitrary Borel subset of $\mathbb{R}^n$. Then the Gaussian measure μ on the cylindrical subsets in the space $MW_2^r(\mathbb{T}^d)$

$$G = \left\{x \in MW_2^r(\mathbb{T}^d) : (\langle x, y_1^{(-r)}\rangle_r, \ldots, \langle x, y_n^{(-r)}\rangle_r) \in B\right\}$$

is given by

$$\mu(G) = \prod_{j=1}^{n}(2\pi\sigma_j)^{-1/2}\int_B \exp\left(-\sum_{j=1}^{n}\frac{|u_j|^2}{2\sigma_j}\right) du_1 \cdots du_n.$$

More detailed information about the Gaussian measure in Banach space is contained in the books of Kuo [12], Ledoux and Talagrand [13].

Here and later on we use the following notations: Assume that $c, c_i, i = 0, 1, \ldots,$ are positive constants depending only on the parameters r, q, ρ and d. For two positive functions $u(y)$ and $v(y)$, $y \in B$, we write $u(y) \asymp v(y)$ or $u(y) \ll v(y)$ if there exist constants c_1 and c_2 or c such that $c_1 \leqslant u(y)/v(y) \leqslant c_2$ or $u(y) \leqslant cv(y)$, $y \in B$.

Now we are in position to state our main results.

Theorem 1. *Let* $r = (r_1, \ldots, r_d)$, $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$, $1 < q < \infty$, $\rho > 1$, *and* $\delta \in (0, 1/2]$. *Then the Probabilistic linear widths of* $W_2^r(\mathbb{T}^d)$ *with the Gaussian measure* μ *in the space* $L_q(\mathbb{T}^d)$ *satisfy asymptotics*

(a) *If* $1 < q \leqslant 2$, *then*

$$\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \asymp (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\left(\ln^{(\nu-1)/2} N\right)\sqrt{1 + (1/N)\ln(1/\delta)}.$$

(b) *If* $2 \leqslant q < \infty$, *then*

$$(N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\left(\ln^{(\nu-1)/q} N\right)\left(1 + N^{-1/q}\sqrt{\ln(1/\delta)}\right) \ll \lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \ll (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\left(\ln^{(\nu-1)/2} N\right)\left(1 + N^{-1/q}\sqrt{\ln(1/\delta)}\right).$$

Remark 1. The order of upper bound in the part (b) of Theorem 1 is different from the lower bound only by a power ln N. We conjecture that the order of upper bound is exact.

Following the method of Maiorov [14] and using Theorem 1, we obtain:

Theorem 2. *Let* $r = (r_1, \ldots, r_d)$, $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$, $1 < q < \infty$, $0 < p < \infty$ *and* $\rho > 1$. *Then the average linear N-width of the space* $MW_2^r(\mathbb{T}^d)$ *in the* $L_q(\mathbb{T}^d)$ *norm has the asymptotic value*

$$\lambda_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \asymp (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\left(\ln^{(\nu-1)/2} N\right).$$

Remark 2. (a) Let $BMW_2^r(\mathbb{T}^d)$ be the unit ball of the multivariate Sobolev space with mixed derivative. The first result on exact order of the classical Kolmogorov N-width of

$BMW_2^r(\mathbb{T}^d)$ in the space $L_q(\mathbb{T}^d)$, $q = 2$ was obtained by Babenko [1], and then the other cases of $1 < q < 2$, and $2 < q < \infty$ were investigated by Galeev [5,6], Temlyakov (see, e.g, [26] for more details). In particular, we have the asymptotic expressions

$$\begin{aligned} d_N\left(BMW_2^r(\mathbb{T}^d),\ L_q(\mathbb{T}^d)\right) &\asymp \left(N^{-1}\ln^{\nu-1} N\right)^{r_1}, \quad 1 < q < \infty, \\ \lambda_N\left(BMW_2^r(\mathbb{T}^d),\ L_q(\mathbb{T}^d)\right) &\asymp \left(N^{-1}\ln^{\nu-1} N\right)^{r_1-(1/2-1/q)_+}, \quad 1 < q < \infty, \end{aligned} \tag{5}$$

where r_j, $j = 1, \ldots, d$, are ordered such that $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$.

(b) It follows from (5), the classical Kolmogorov and linear N-widths are equal modulo multiplicative constants for $BMW_2^r(\mathbb{T}^d)$ in the space $L_q(\mathbb{T}^d)$, $1 < q \leqslant 2$; however, for $q > 2$ $d_N\left(MW_2^r(\mathbb{T}^d),\ L_q(\mathbb{T}^d)\right)$ is essentially less than the linear width $\lambda_N\left(MW_2^r(\mathbb{T}^d),\right.$ $\left.L_q(\mathbb{T}^d)\right)$. The linear operators lose to optimal nonlinear operators by a factor $N^{1/2-1/q}$.

In [2], we have proved that under the condition of Theorem 1, the average Kolmogorov N-widths of $MW_2^r(\mathbb{T}^d)$ with the Gaussian measure μ in the space $L_q(\mathbb{T}^d)$ satisfy asymptotic relation

$$d_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \asymp (N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2}\,(\ln^{(\nu-1)/2} N). \tag{6}$$

Comparing this with Theorem 2, it is interesting to note that in the average case setting, the Kolmogorov N-width and linear N-width of $MW_2^r(\mathbb{T}^d)$ in the $L_q(\mathbb{T}^d)$ space, $1 < q < \infty$, have the same error order. This means that for most functions in class of $MW_2^r(\mathbb{T}^d)$, the optimal linear operators are (modulo a constant) as good as nonlinear operators.

Remark 3. (a) In the case of one dimension $d = 1$, $1 \leqslant q \leqslant \infty$, Theorems 1 and 2 were proved by Maiorov [15], and Fang and Ye [3,4].

(b) If the dimension $d \geqslant 2$, then the asymptotic exact order of Probabilistic linear widths of $W_2^r(\mathbb{T}^d)$ with the Gaussian measure μ in the $L_q(\mathbb{T}^d)$ space for the cases $q = 1, 2 < q \leqslant \infty$; and the average linear widths $\lambda_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)$ for the case $q = 1$ and $q = \infty$ are still open.

The organization of the paper is as follows. In Section 2, we establish two discretization theorems, which will be used to show the upper and lower estimates of Theorem 1, respectively. In Section 3, we give the proof of Theorems 1 and 2.

2. Discretization

In order to prove Theorem 1, we use the discretization method (see [9,14]), which is based on the reduction of the calculation of the probabilistic linear widths of a given class to the computation of widths of finite-dimensional set equipped with the standard Gaussian measure. First, we recall the definitions and cite some results on the linear (N, δ)-widths of finite-dimensional set, which play an important roles in the proof of Theorem 1. Let ℓ_p^m

be m-dimensional normed space of vectors $x = (x_1 \dots x_m) \in \mathbb{R}^m$, with a norm

$$\|x\|_{\ell_p^m} = \begin{cases} \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}, & 1 \leqslant p < \infty, \\ \max_{1 \leqslant i \leqslant m} |x_i|, & p = \infty. \end{cases}$$

Consider in $\mathbb{R}^m$ the standard Gaussian measure $\upsilon = \upsilon_m$, which is defined as

$$\upsilon(G) = (2\pi)^{-m/2} \int_G \exp\left(-\frac{1}{2}\|x\|_2^2\right) dx,$$

where G is any Borel subset in $\mathbb{R}^m$. Obviously, $\upsilon(\mathbb{R}^m) = 1$.

Let $N = 0, 1, \dots$, and $\delta \in [0, 1)$ be arbitrary. We define linear (N, δ)-width of the space $\mathbb{R}^m$ equipped with the standard Gaussian measure υ in ℓ_q^m-norm:

$$\lambda_{N,\delta}(\mathbb{R}^m, \upsilon, \ell_q^m) = \inf_{G_\delta} \inf_{T_N} \sup_{x \in \mathbb{R}^m \setminus G_\delta} \|x - T_N x\|_{\ell_q^m},$$

where T_N runs over all linear operator from X to X with rank at most N.

The following two lemmas will be used in the proof of Theorem 1.

Lemma 1 (*Maiorov [15]*). *If* $1 \leqslant q \leqslant 2$, $m \geqslant 2N$ *and* $\delta \in (0, 1/2]$, *then*

$$\lambda_{N,\delta}(\mathbb{R}^m, \upsilon, \ell_q^m) \asymp m^{1/q} + \sqrt{\ln(1/\delta)}.$$

Lemma 2 (*Gensun and Peixin [3]*). *If* $1 \leqslant q \leqslant 2$, $m \geqslant 2N$ *and* $\delta \in (0, 1/2]$, *then*

$$\lambda_{N,\delta}(\mathbb{R}^m, \upsilon, \ell_q^m) \asymp m^{1/q-1/2} \sqrt{m + \ln(1/\delta)}$$

and if $m > N$, *then*

$$\lambda_{N,\delta}(\mathbb{R}^m, \upsilon, \ell_q^m) \ll m^{1/q-1/2} \sqrt{m + \ln(1/\delta)}.$$

We now start to establish the discretization theorem. First, we introduce some notations and lemmas. It is convenient in many cases to split the Fourier series of a function into the sum of diadic blocks. We associate every vector $s = (s_1, \dots, s_d) \in \mathbb{N}^d$ whose coordinates are natural numbers with the set

$$\square_s = \{n = (n_1, \dots, n_d) \in \mathbb{Z}_0^d : \ 2^{s_j - 1} \leqslant |n_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

and let $x_s(\cdot)$ denote the "block" of the Fourier series for $x(\cdot)$, namely

$$\delta_s x(\cdot) := x_s(\cdot) := \sum_{n \in \square_s} c_n e^{i(n,\cdot)}.$$

The next two known lemmas are crucial for establishing discretization theorem (see Theorems 3 and 4).

Lemma 3 (*Galeev [5]*). *Let S be a subset of* $\mathbb{N}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$, $1 < q < \infty$ *and* $x = \sum_{s\in S} \delta_s x \in F_s$. *Then we have*

$$|S|^{(1/2-1/q)_-} \left(\sum_{s\in S} \|2^{(s,\alpha)} \delta_s x\|^q_{L^d_q} \right)^{1/q} \ll \|x^{(\alpha)}\|_{L^d_q} \ll |S|^{(1/2-1/q)_+} \left(\sum_{s\in S} \|2^{(s,\alpha)} \delta_s x\|^q_{L^d_q} \right)^{1/q}, \tag{7}$$

where $a_- = \min\{0, a\}$, $b_+ = \max\{0, b\}$, $|S|$ *denotes the cardinality of the set S, and*

$$F_S = \operatorname{span}\{e^{i(n,\cdot)} : n \in \square_s,\ s \in S\}.$$

Lemma 4 (*Galeev [5]*). *Let* $s \in \mathbb{N}^d$. *Then the space of trigonometric polynomials*

$$\operatorname{span}\left\{e^{i(n,\cdot)} : n \in \square_s\right\}$$

is isomorphic to the space $\mathbb{R}^{2^{(s,1)}}$ *via mapping*

$$x(t) \longmapsto \{x_{s,m}(t_j)\}_{m,j}, \qquad x_{s,m}(t) = \sum_{\substack{n\in\square_s \\ \operatorname{sgn} n = \operatorname{sgn} m}} c_n e^{i(n,t)}$$

$$m = (m_1, \dots, m_d) = (\pm 1, \dots, \pm 1) \in \mathbb{R}^d,\quad t_j = (\pi 2^{2-s_1} j_1, \dots, \pi 2^{2-s_d} j_d) \in \mathbb{R}^d,$$
$$j_i = 1, \dots, 2^{s_i - 1}, \qquad i = 1, \dots d.$$

Moreover, the following relation is true:

$$\|x\|_{L^q_d} \asymp 2^{-(s,1/q)} \|\{x_{s,m}(t_j)\}_{m,j}\|_{\ell_q^{2(s,1)}}, \qquad 1 < q < \infty, \tag{8}$$

where in the equivalence norm (8), *the constants do not depend on s*.

Now we are ready to establish a discretization theorem which reduces the computation of the upper bound for probabilistic linear (N, δ)-width $\lambda_{N,\delta}(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d))$ to the corresponding finite-dimensional problem for the linear (N, δ)-width $\lambda_N(\mathbb{R}^m, \upsilon, \ell_q^m)$. Below, we always assume that

$$\gamma = r + \rho/2, \qquad \gamma' = \gamma/(r_1 + \rho/2)$$

and for natural numbers k and ℓ, let

$$S_{\ell,k} = \{s \in \mathbb{N}^d : \ell - 1 \leqslant (s, \gamma') < \ell, (s, 1) = k\}, \tag{9}$$

i.e.,

$$S_{\ell,k} = \left\{ s \in \mathbb{N}^d : \sum_{i=1}^d s_i = k, \ell - 1 \leqslant k\gamma' < \ell \right\},$$

and let $\|S_{\ell,k}\| = \sum_{s\in S_{\ell,k}} |\Box_s|$. Then we have $\|S_{\ell,k}\| = 2^k|S_{\ell,k}|$. It is clear, we always have $k \geqslant d$ and $S_{\ell,k} = \phi$ if $k \geqslant \ell$. Set

$$F_{\ell,k} = \mathrm{span}\{e^{i(n,\cdot)} : n \in \Box_s, \ s \in S_{\ell,k}\}.$$

Theorem 3. *Let* $1 < q < \infty$, $r = (r_1, \dots, r_d) \in \mathbb{R}^d$, $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$, $N = 0, 1, \dots$, $\delta \in (0, 1/2]$ *and let the sequences of numbers* $\{N_{\ell,k}\}$ *and* $\{\delta_{\ell,k}\}$ *be such that* $0 \leqslant N_{\ell,k} \leqslant \|S_{\ell,k}\|$, $\sum_{\ell,k} N_{\ell,k} \leqslant N$, *and* $\sum_{\ell,k} \delta_{\ell,k} \leqslant \delta$. *Then*

$$\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)\right) \ll \sum_{\ell,k} 2^{-(r_1+\rho/2)\ell+k/2-k/q}|S_{\ell,k}|^{(1/2-1/q)_+}\lambda_{N_{\ell,k},\delta_{\ell,k}}\left(\mathbb{R}^{\|S_{\ell,k}\|}, \upsilon, \ell_q^{\|S_{\ell,k}\|}\right).$$

Proof. It follows from Lemma 3 that

$$|S_{\ell,k}|^{(1/2-1/q)_-}\left(\sum_{s\in S_{\ell,k}} \|\delta_s x\|_{L_q^d}^q\right)^{1/q} \ll \|x\|_{L_q^d} \ll \left|S_{\ell,k}\right|^{(1/2-1/q)_+}\left(\sum_{s\in S_{\ell,k}} \|\delta_s x\|_{L_q^d}^q\right)^{1/q}, \tag{10}$$

where $x \in F_{\ell,k}$. Note that $\gamma = r + \rho/2$ and $(s, 1) = k$ for $s \in S_{\ell,k}$, then the definition of $\delta_s x$ implies

$$\begin{aligned}\left(\sum_{s\in S_{\ell,k}} \|\delta_s x^{(r)}\|_{L_q^d}^q\right)^{1/q} &\asymp \left(\sum_{s\in S_{\ell,k}} \|2^{(s,r)}\delta_s x\|_{L_q^d}^q\right)^{1/q} \\ &= \left(\sum_{s\in S_{\ell,k}} 2^{q(s,r+\rho/2)-q(s,\rho/2)}\|\delta_s x\|_{L_q^d}^q\right)^{1/q} \\ &= \left(\sum_{s\in S_{\ell,k}} 2^{q(s,\gamma)-qk\rho/2}\|\delta_s x\|_{L_q^d}^q\right)^{1/q} \\ &\asymp 2^{(r_1+\rho/2)\ell}\left(\sum_{s\in S_{\ell,k}} 2^{-qk\rho/2}\|\delta_s\ x\|_{L_q^d}^q\right)^{1/q} \\ &= 2^{(r_1+\rho/2)\ell-k\rho/2}\left(\sum_{s\in S_{\ell,k}} \|\delta_s\ x\|_{L_q^d}^q\right)^{1/q}.\end{aligned}$$

Substituting the above relation in (10), we get

$$|S_{\ell,k}|^{(1/2-1/q)_-}2^{-(r_1+\rho/2)\ell+k\rho/2}\left(\sum_{s\in S_{\ell,k}}\|\delta_s x^{(r)}\|^q_{L^d_q}\right)^{1/q}$$
$$\ll \|x\|_{L^d_q} \ll |S_{\ell,k}|^{(1/2-1/q)_+}2^{-(r_1+\rho)\ell+k\rho/2}\left(\sum_{s\in S_{\ell,k}}\|\delta_s x^{(r)}\|^q_{L^d_q}\right)^{1/q},$$

which together with Eq. (8) in Lemma 4 for the $\delta_s x^{(r)}$ implies

$$|S_{\ell,k}|^{(1/2-1/q)_-}2^{-(r_1+\rho/2)\ell+k\rho/2}\left(\sum_{s\in S_{\ell,k}}2^{-(s,1)}\|\{(\delta_s x^{(r)})(t_j)\}_{m,j}\|^q_{\ell_q^{2^{(s,1)}}}\right)^{1/q}$$
$$\ll \|x\|_{L^d_q} \ll |S_{\ell,k}|^{(1/2-1/q)_+}2^{-(r_1+\rho/2)\ell+k\rho/2}$$
$$\times\left(\sum_{s\in S_{\ell,k}}2^{-(s,1)}\|\{(\delta_s x^{(r)})(t_j)\}_{m,j}\|^q_{\ell_q^{2^{(s,1)}}}\right)^{1/q}. \tag{11}$$

Now we consider in the space $F_{\ell,k}$ the polynomials

$$\varphi^{\ell,k}_{s,m,j}(t)=\sum_{\substack{n\in\square_s\\ \operatorname{sign} n=\operatorname{sign} m}} e_n(t-t_j),\qquad s\in S_{\ell,k},$$

$$m=(m_1,\ldots,m_d)=(\pm1,\ldots,\pm1)\in\mathbb{R}^d,\quad j_i=1,\ldots,2^{s_i-1},\qquad i=1,\ldots d.$$

Obviously, these polynomials are orthogonal in $L_2(\mathbb{T}^d)$, and for any $x\in F_{\ell,k}$,

$$(D^r x)_{s,m}(t_j)=\langle D^r x,\varphi^{\ell,k}_{s,m,j}\rangle,\qquad \forall s,\ m,\ j.$$

Plugging this into (11), and noting that $(s,1)=k$ for any $s\in S_{\ell,k}$, we get

$$|S_{\ell,k}|^{(1/2-1/q)_-}2^{-(r_1+\rho/2)\ell+k\rho/2-k/q}\|\{\langle D^r x,\varphi^{\ell,k}_{s,m,j}\rangle\}_{s,m,j}\|_{\ell_q^{\|S_{\ell,k}\|}}$$
$$\ll \|x\|_{L^d_q} \ll |S_{\ell,k}|^{(1/2-1/q)_+}2^{-(r_1+\rho/2)\ell+k\rho/2-k/q}$$
$$\times\|\{\langle D^r x,\varphi^{\ell,k}_{s,m,j}\rangle\}_{s,m,j}\|_{\ell_q^{\|S_{\ell,k}\|}}. \tag{12}$$

Now for any $\ell,\ k\in\mathbb{N}$ and $d\leqslant k\leqslant\ell$, we consider a mapping

$$I_{\ell,k}:F_{\ell,k}\to\ell_q^{\|S_{\ell,k}\|}\qquad x\mapsto\{\langle D^r x,\varphi^{\ell,k}_{s,m,j}\rangle\}_{s,m,j}.$$

It follows from (12) and Lemma 4 that $I_{\ell,k}$ is linear isomorphic from the space $F_{\ell,k}$ to the space $\ell_q^{\|S_{\ell,k}\|}$.

In the sequel for convenience, we write

$$\lambda_{N_{\ell,k},\delta_{\ell,k}}:=\lambda_{N_{\ell,k},\delta_{\ell,k}}\left(\mathbb{R}^{\|S_{\ell,k}\|},\upsilon,\ell_q^{\|S_{\ell,k}\|}\right).$$

Denote by $L_{\ell,k}$ a linear operator from $\mathbb{R}^{\|S_{\ell,k}\|}$ to $\mathbb{R}^{\|S_{\ell,k}\|}$ such that dim $L_{\ell,k} \leqslant N_{\ell,k}$ and

$$\upsilon\left\{ y \in \mathbb{R}^{\|S_{\ell,k}\|} : \|y - L_{\ell,k}y\|_{\ell_q^m} > \lambda_{N_{\ell,k},\delta_{\ell,k}} \right\} \leqslant \delta_{\ell,k}. \tag{13}$$

For any $x \in MW_2^r(\mathbb{T}^d)$, let $\Delta_{\ell,k}x = \sum_{s\in S_{\ell,k}} \delta_s x$. Then by virtue of (12) there exists constant c_1 independent of ℓ and k such that

$$\begin{aligned}
&\|\Delta_{\ell,k}x - D^{-r} I_{\ell,k}^{-1} L_{\ell,k} I_{\ell,k} \Delta_{\ell,k}x\|_{L_q^d}\\
&\quad\leqslant c_1 |S_{\ell,k}|^{(1/2-1/q)_+} 2^{-(r_1+\rho/2)\ell + k\rho/2 - k/q}\\
&\qquad\times \left\| \{\langle D^r x, \varphi_{s,m,j}^{\ell,k}\rangle\}_{s,m,j} - L_{\ell,k}\{\langle D^r x, \varphi_{s,m,j}^{\ell,k}\rangle\}_{s,m,j} \right\|_{\ell_q^{\|S_{\ell,k}\|}}.
\end{aligned} \tag{14}$$

Set $\sigma_{s,m,j}^{\ell,k} := \langle C_\mu \varphi_{s,m,j}^{\ell,k}, \varphi_{s,m,j}^{\ell,k}\rangle$. Then it is clear that all of $\sigma_{s,m,k}^{\ell,k}$ are equal and

$$\begin{aligned}
\sigma_{s,m,j}^{\ell,k} &= \sum_{n\in\mathbb{Z}_0^d} |n|^{-\rho} |\langle \varphi_{s,m,j}^{\ell,k}, e^{i(n,\cdot)}\rangle|^2\\
&= \sum_{\substack{n\in\square_s,\\ \operatorname{sign} n = \operatorname{sign} m}} |n|^{-\rho} \asymp (2^{(s,1)})^{(1-\rho)} = 2^{-k(\rho-1)}.
\end{aligned}$$

Therefore there exists a constant c_2 such that

$$\sigma := \sigma_{s,m,j}^{\ell,k} = c_2^2 2^{-k(\rho-1)}, \qquad \forall s, m, j. \tag{15}$$

Consider the set of $MW_2^r(\mathbb{T}^d)$

$$\begin{aligned}
G_{\ell,k} = \{&x \in MW_2^r(\mathbb{T}^d) : \|\Delta_{\ell,k}x - D^{-r} I_{\ell,k}^{-1} L_{\ell,k} \Delta_{\ell,k}x\|_{L_q^d}\\
&> c_1 c_2 |S_{\ell,k}|^{(1/2-1/q)_+} 2^{-(r_1+\rho/2)\ell + k/2 - k/q} \lambda_{N_{\ell,k},\delta_{\ell,k}}\}.
\end{aligned}$$

From (14), the definitions of the measure μ and the standard Gaussian measure υ in the space $\mathbb{R}^{\|S_{\ell,k}\|}$ and (13),

$$\begin{aligned}
\mu(G_{\ell,k}) &\leqslant \mu\Big\{x \in MW_2^r(\mathbb{T}^d) : \|\Delta_{\ell,k}x - D^{-r} I_{\ell,k}^{-1} L_{\ell,k} I_{\ell,k} \Delta_{\ell,k}x\|_{L_q^d}\\
&\qquad > \sigma^{1/2} \lambda_{N_{\ell,k},\delta_{\ell,k}}\Big\}\\
&= \upsilon\left\{y \in \mathbb{R}^{\|S_{\ell,k}\|} : \|y\sigma^{1/2} - L_{\ell,k} y \sigma^{1/2}\|_{\ell_q^{\|S_{\ell,k}\|}} > \sigma^{1/2}\lambda_{N_{\ell,k},\delta_{\ell,k}}\right\}\\
&= \upsilon\left\{y \in \mathbb{R}^{\|S_{\ell,k}\|} : \|y - L_{\ell,k} y\|_{\ell_q^{\|S_{\ell,k}\|}} > \lambda_{N_{\ell,k},\delta_{\ell,k}}\right\} \leqslant \delta_{\ell,k}.
\end{aligned} \tag{16}$$

Let us consider the set $G = \cup_{\ell,k} G_{\ell,k}$ and the linear operator T_N defined from $MW_2^r(\mathbb{T}^d)$ to $MW_2^r(\mathbb{T}^d)$, which is $T_N x = \sum_{\ell,k} D^{-r} I_{\ell,k}^{-1} L_{\ell,k} I_{\ell,k} \Delta_{\ell,k} x$. From the hypothesis of the theorem, we get

$$\mu(G) \leqslant \sum_{\ell,k} \mu(G_{\ell,k}) \leqslant \sum_{\ell,k} \delta_{\ell,k} \leqslant \delta$$

and

$$\dim\, T_N \leqslant \sum_{\ell,k} \dim\, L_{\ell,k} \leqslant \sum_{\ell,k} N_{\ell,k} \leqslant N.$$

Consequently, by the definitions of G, T_N, $\{G_{\ell,k}\}$ and $\{L_{\ell,k}\}$,

$$\begin{aligned} &\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)\right) \\ &\quad\leqslant \sup_{x\in MW_2^r(\mathbb{T}^d)\setminus G} \|x - T_N x\|_{L_q^d} \\ &\quad\leqslant \sup_{x\in MW_2^r(\mathbb{T}^d)\setminus G} \sum_{\ell,k} \|\Delta_{\ell,k}x - D^{-r} I_{\ell,k}^{-1} L_{\ell,k} I_{\ell,k} \Delta_{\ell,k} x\|_{L_q^d} \\ &\quad\ll \sum_{\ell,k} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{(1/2-1/q)_+} \lambda_{N_{\ell,k},\delta_{\ell,k}}\left(\mathbb{R}^{\|S_{\ell,k}\|}, \upsilon, \ell_q^{\|S_{\ell,k}\|}\right), \end{aligned}$$

which complete the proof of Theorem 3. □

To establish the upper bound of Theorem 1, we also need the following lemma.

Lemma 5 (*Romanyuk [25]*). *Let $N \in \mathbb{N}$, $N \asymp 2^u u^{v-1}$, $\beta > 0$, $S_{\ell,k}$ be defined by* (9) *and*

$$N_{\ell,k} := \begin{cases} \|S_{\ell,k}\|, & d \leqslant k \leqslant \ell,\ \ell \leqslant u, \\ \lfloor |S_{\ell,k}| 2^{u+\beta u-2\beta\ell+\beta k} \rfloor, & d \leqslant k \leqslant \ell,\ \ \ell > u, \\ 0 & \textit{others}. \end{cases}$$

Then

$$\sum_{\ell,k} N_{\ell,k} \ll N, \tag{17}$$

where $\lfloor a \rfloor$ means the largest integer no greater than a.

We suppose that in Lemma 5 the constant $\beta > 0$ satisfies also the condition

$$0 < \beta < \min\{2r_1 + \rho - 2,\ 1/2\} \tag{18}$$

and will be selected in the course of establishing the required upper bound of Theorem 1.

To proceed the lower estimate of Theorem 1, we prove another discretization theorem, which reduces the computation of the lower bound for probabilistic linear (N, δ)-width to the estimate of the lower bound on finite-dimensional problem for the (N, δ)-width $\lambda_N(\mathbb{R}^m, \upsilon, \ell_q^m)$. Thus in a certain sense, it is a converse of Theorem 3. First we give some notations.

Let $S = S_{k_0} = \{s = (s_1, \ldots, s_v, 1, \ldots, 1) \in \mathbb{N}^d :\ (s, 1) = k = [k_0]\}$, where k_0 will be chosen later. It is not difficult to prove that $|S| \asymp k^{v-1}$, that is, there exist two positive constants c_3 and c_4 such that $c_3 k^{v-1} \leqslant |S| \leqslant c_4 k^{v-1}$. We choose k_0 such that the number of harmonics in the set of S is at least $2N$, i.e.,

$$\|S\| := \sum_{s\in S} |\Box_s| = \sum_{s\in S} 2^{(s,1)} = |S| 2^k \geqslant c_3 k^{v-1} 2^k \geqslant c_5 k_0^{v-1} 2^{k_0} = 2N$$

and

$$2^k k^{\nu-1} \asymp N \asymp |S| 2^k. \tag{19}$$

We consider the space of trigonometric polynomials

$$F_S = \operatorname{span}\{e^{i(n,t)} : n \in \square_s, s \in S\}.$$

In the proof of (12), if we let $S_{\ell,k} = S$, and note that $(s,1) = k$ for any $s \in S$, it follows that there exist two positive constants c_6 and c_7 such that

$$\begin{aligned} &c_6 |S|^{(1/2-1/q)_-} 2^{-(r_1+1/q)k} \|\{\langle D^r x, \varphi^{k,k}_{s,m,j}\rangle\}_{s,m,j}\|_{\ell_q^{\|S\|}} \\ &\quad \leqslant \|x\|_{L_q^d} \leqslant c_7 |S|^{(1/2-1/q)_+} 2^{-(r_1+1/q)k} \|\{\langle D^r x, \varphi^{k,k}_{s,m,j}\rangle\}_{s,m,j}\|_{\ell_q^{\|S\|}}. \end{aligned} \tag{20}$$

Let

$$I_S : \ F_S \to \ell_q^{\|S\|}, \quad x \mapsto \{\langle D^r x, \varphi^{k,k}_{s,m,j}\rangle\}_{s,m,j}.$$

Then by virtue of Lemma 4 and (20), I_S is a linear isomorphic mapping from the space of trigonometric polynomials F_S to $\ell_q^{\|S\|}$.

Now, we are ready to prove the another discretization theorem.

Theorem 4. *Suppose that* $1 < q < \infty$, $r = (r_1, \dots, r_d) \in \mathbb{R}^d$, $1/2 < r_1 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$, $N = 0, 1, \dots$, $\delta \in (0, 1/2]$. *Then it follows that*

$$\begin{aligned} \lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \gg\ & 2^{-(r_1+1/q+(\rho-1)/2)k} |S|^{(1/2-1/q)_-} \\ & \times \lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},\ \upsilon,\ \ell_q^{\|S\|}\right), \end{aligned}$$

where the set S is defined by (19).

Proof. Let T_1 be a linear operator from $MW_q(\mathbb{T}^d) \cap F_S$ to $MW_2(\mathbb{T}^d) \cap F_S$ such that $\dim T_1 \leqslant N$ and

$$\mu\left\{x \in MW_2^r(\mathbb{T}^d) \cap F_S : \|x - T_1 x\|_{L_q} > \lambda_{N,\delta}\right\} \leqslant \delta, \tag{21}$$

where $\lambda_{N,\delta} := \lambda_{N,\delta}(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d))$. Set

$$\begin{aligned} G = \Big\{ & y \in \mathbb{R}^{\|S\|} : \|y - I_S T_1 D^r I_S^{-1} y\|_{L_q^d} \\ & > c_2^{-1} c_6 2^{(r_1+1/q+(\rho-1)/2)k} |S|^{(1/q-1/2)_-} \lambda_{N,\delta} \Big\}, \end{aligned}$$

where c_2, c_6 are defined by (15), (20), respectively. Then from (20) and (21), we get

$$\begin{aligned} \upsilon(G) \leqslant \upsilon \Big\{ & y \in \mathbb{R}^{\|S\|} : \|y\sigma^{-1/2} - I_S T_1 D^r I_S^{-1} y \sigma^{-1/2}\|_{\ell_q^{\|S\|}} \\ & > c_6 2^{(r_1+1/q)k} |S|^{(1/q-1/2)_-} \lambda_{N,\delta} \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \mu \Big\{ x \in MW_2^r(\mathbb{T}^d) \cap F_S : \|\{\langle D^r x, \varphi_{s,m,j}^{k,k}\rangle\}_{s,m,j} \\
&\quad - I_S T_1 D^r I_S^{-1} \{\langle D^r x, \varphi_{s,m,j}^{k,k}\rangle\}_{s,m,j}\|_{\ell_q^{\|S\|}} \\
&\quad > c_6 2^{(r_1+1/q)k} |S|^{(1/q-1/2)_-} \lambda_{N,\delta} \Big\} \\
&\leqslant \mu \Big\{ x \in MW_2^r(\mathbb{T}^d) \cap F_S : \|x - T_1 x\|_{L_q^d} > \lambda_{N,\delta} \Big\} \leqslant \delta, \qquad (22)
\end{aligned}
$$

where the constant σ is defined by (15). Clearly, $\dim\{I_S T_1 D^r I_S^{-1}\} \leqslant N$. Therefore, by (12),

$$
\begin{aligned}
\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},\ \upsilon,\ \ell_q^{\|S\|}\right) &\leqslant \sup_{y \in \mathbb{R}^{\|S\|} \setminus G} \|y - I_S T_1 D^r I_S^{-1} y\|_{\ell_q^{\|S\|}} \\
&\ll 2^{(r_1+1/q+(\rho-1)/2)k} |S|^{(1/q-1/2)_-} \lambda_{N,\delta}.
\end{aligned}
$$

That is

$$
\begin{aligned}
\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)\right) &\gg 2^{-(r_1+1/q+(\rho-1)/2)k} |S|^{(1/2-1/q)_-} \\
&\quad \times \lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},\ \upsilon,\ \ell_q^{\|S\|}\right), \qquad (23)
\end{aligned}
$$

which completes the proof of Theorem 4. □

3. Proofs of main results

We are in a position to prove Theorem 1 which is the main result of this paper.

Proof of Theorem 1. We begin with the upper bound. It is clear that we only need to prove the upper estimate for the case $2 \leqslant q < \infty$. Choose a constant $0 < \beta < 1/2$, and for given $N \in \mathbb{N}$, select a u according to the condition $N \asymp 2^u u^{\nu-1}$. We define $N_{\ell,k}$ as in Lemma 5, and let

$$
\delta_{\ell,k} = \begin{cases} \delta N_{\ell,k}/N, & d \leqslant k \leqslant \ell,\ \ell > u, \\ 0 & \text{others.} \end{cases}
$$

From the definition of $\delta_{\ell,k}$ and (17), we get

$$
\sum_{\ell,k} \delta_{\ell,k} \ll \delta. \qquad (24)
$$

By virtue of (17) and (24), we know that $\{N_{\ell,k}\}$ and $\{\delta_{\ell,k}\}$ satisfy the conditions in Theorem 3. By Theorem 3 and Lemma 1, we have

$$
\begin{aligned}
&\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)\right) \\
&\quad \ll \sum_{\ell,k} 2^{-(r_1+\rho/2)\ell + k/2 - k/q} |S_{\ell,k}|^{1/2-1/q} \lambda_{N_{\ell,k},\delta_{\ell,k}}\left(\mathbb{R}^{\|S_{\ell,k}\|}, \upsilon, \ell_q^{\|S_{\ell,k}\|}\right) \\
&\quad \ll \sum_{\ell > u} \sum_{d \leqslant k \leqslant \ell} 2^{-(r_1+\rho/2)\ell + k/2 - k/q} |S_{\ell,k}|^{1/2-1/q} \lambda_{N_{\ell,k},\delta_{\ell,k}}\left(\mathbb{R}^{\|S_{\ell,k}\|}, \upsilon, \ell_q^{\|S_{\ell,k}\|}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\ll \sum_{\ell>u} \sum_{d\leqslant k\leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{1/2-1/q} \left(\|S_{\ell,k}\|^{1/q} + \sqrt{\ln(N/(N_{\ell,k}\delta))} \right) \\
&\ll \sum_{\ell>u} \sum_{d\leqslant k\leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{1/2-1/q} \|S_{\ell,k}\|^{1/q} \\
&\quad + \sum_{\ell>u} \sum_{d\leqslant k\leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{1/2-1/q} N_{\ell,k}^{-1/2} N^{1/2} \\
&\quad + \sum_{\ell>u} \sum_{d\leqslant k\leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{1/2-1/q} \sqrt{\ln(1/\delta)} \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{25}
$$

In the course of the proof of the second inequality, we have used a simple fact

$$
\lambda_{N_{\ell,k},\delta_{\ell,k}} \left(\mathbb{R}^{\|S_{\ell,k}\|}, \upsilon, \ell_q^{\|S_{\ell,k}\|} \right) = 0, \qquad d \leqslant k \leqslant u \text{ or } k < d \text{ or } k > \ell.
$$

Our next aim is to estimate the three terms at the end of expression (25). Since $r_1 > 1/2$, we can choose a constant β such that the condition $0 < \beta < 1/2$ is satisfied. We start with the term I_1,

$$
\begin{aligned}
I_1 &= \sum_{\ell>u} \sum_{d\leqslant k\leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q} |S_{\ell,k}|^{1/2-1/q} |S_{\ell,k}|^{1/q} 2^{k/q} \\
&= \sum_{\ell>u} 2^{-(r_1+\rho/2)\ell} \sum_{d\leqslant k\leqslant \ell} 2^{k/2} |S_{\ell,k}|^{1/2}.
\end{aligned} \tag{26}
$$

Now we begin to deal with the inner sum in (26). For this purpose, using an idea of [25], we represent this sum in the form

$$
\sum_{d\leqslant k\leqslant \ell} 2^{k/2} |S_{\ell,k}|^{1/2} = \left[\sum_{d\leqslant k\leqslant \ell}{}' + \sum_{d\leqslant k\leqslant \ell}{}'' \right] 2^{k/2} |S_{\ell,k}|^{1/2}, \tag{27}
$$

where the summation in $\sum'_{d\leqslant k\leqslant \ell}$ is carried out over k with $|S_{\ell,k}| \leqslant \ell^{\nu-1}$, and the summation in $\sum''_{d\leqslant k\leqslant \ell}$ is carried out over k for which $|S_{\ell,k}| > \ell^{\nu-1}$. We have

$$
\sum_{d\leqslant k\leqslant \ell}{}' 2^{k/2} |S_{\ell,k}|^{1/2} \leqslant \ell^{(\nu-1)/2} \sum_{d\leqslant k\leqslant \ell}{}' 2^{k/2} \ll \ell^{(\nu-1)/2} 2^{\ell/2} \tag{28}
$$

and

$$
\begin{aligned}
\sum_{d\leqslant k\leqslant \ell}{}'' 2^{k/2} |S_{\ell,k}|^{1/2} &= \sum_{d\leqslant k\leqslant \ell}{}'' 2^{k/2} |S_{\ell,k}|\, |S_{\ell,k}|^{-1/2} \\
&\ll \ell^{-(\nu-1)/2} \sum_{(s,\gamma')\leqslant \ell} 2^{k/2(s,1)} \ll \ell^{-(\nu-1)/2} 2^{\ell/2} \ell^{\nu-1} \\
&= 2^{\ell/2} \ell^{(\nu-1)/2}.
\end{aligned} \tag{29}
$$

Thus, substituting (28) and (29) in (27), we obtain

$$\sum_{d \leqslant k \leqslant \ell} 2^{k/2}|S_{\ell,k}|^{1/2} \ll 2^{\ell/2}\ell^{(v-1)/2}.$$

Return to (26). It follows that

$$\begin{aligned} I_1 &\ll \sum_{\ell>u} 2^{-(r_1+(\rho-1)/2)\ell}\ell^{(v-1)/2} \ll 2^{-(r_1+(\rho-1)/2)u}u^{(v-1)/2} \\ &\asymp (N^{-1}\ln^{v-1}N)^{r_1+(\rho-1)/2}(\ln^{(v-1)/2}N). \end{aligned} \tag{30}$$

Next we estimate the term I_2. Using the condition $0<\beta<1/2$, we get

$$\begin{aligned} I_2 &= \sum_{\ell>u}\sum_{d \leqslant k \leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q}|S_{\ell,k}|^{1/2-1/q}|S_{\ell,k}|^{-1/2} \\ &\quad \times 2^{-u/2-\beta u/2+\beta\ell-\beta k/2}N^{1/2} \\ &\ll N^{1/2}2^{-u/2-\beta u/2}\sum_{\ell>u}2^{-(r_1+\rho/2)\ell+\beta\ell}\sum_{d \leqslant k \leqslant \ell}2^{(1/2-\beta/2-1/q)k}|S_{\ell,k}|^{-1/q} \\ &\ll N^{1/2}2^{-u/2-\beta u/2}\sum_{\ell>u}2^{-(r_1+\rho/2)\ell+\beta\ell}\sum_{d \leqslant k \leqslant \ell}2^{(1/2-\beta/2)k} \\ &\ll N^{1/2}2^{-u/2-\beta u/2}\sum_{\ell>u}2^{-(r_1+\rho/2-1/2-\beta/2)\ell} \\ &\ll N^{1/2}2^{-u/2-\beta u/2}2^{-(r_1+\rho/2-1/2-\beta/2)u} \\ &= N^{1/2}2^{-(r_1+(\rho-1)/2)u}2^{-u/2} \asymp u^{(v-1)/2}2^{u/2}2^{-(r_1+(\rho-1)/2)u}2^{-u/2} \\ &= 2^{u/2}2^{-(r_1+(\rho-1)/2)u}u^{(v-1)/2} \asymp (N^{-1}\ln^{v-1}N)^{r_1+(\rho-1)/2}(\ln^{(v-1)/2}N). \end{aligned} \tag{31}$$

Finally, we proceed the term I_3. Using the condition $0<\beta<2r_1+\rho-2$ (see (18)), we derive

$$\begin{aligned} I_3 &= \sum_{\ell>u}\sum_{d \leqslant k \leqslant \ell} 2^{-(r_1+\rho/2)\ell+k/2-k/q}|S_{\ell,k}|^{1/2-1/q}\sqrt{\ln(1/\delta)} \\ &\ll \sum_{\ell>u}2^{-(r_1+\rho/2)\ell}\sum_{d \leqslant k \leqslant \ell}2^{(1/2-1/q)k}|S_{\ell,k}|^{1/2-1/q}\sqrt{\ln(1/\delta)}. \end{aligned} \tag{32}$$

Using the method of computing I_1, we get

$$\sum_{d \leqslant k \leqslant \ell} 2^{(1/2-1/q)k}|S_{\ell,k}|^{1/2-1/q} \ll 2^{(1/2-1/q)\ell}\ell^{1/2-1/q}.$$

Substituting above inequality in (32), we have

$$\begin{aligned} I_3 &\ll \sum_{\ell>u}2^{-(r_1+(\rho-1)/2+1/q)\ell}\left(\ell^{v-1}\right)^{1/2-1/q}\sqrt{\ln(1/\delta)} \\ &\ll 2^{-(r_1+(\rho-1)/2)u}u^{(v-1)/2}\left(2^u u^{v-1}\right)^{-1/q}\sqrt{\ln(1/\delta)} \\ &\asymp (N^{-1}\ln^{v-1}N)^{r_1+(\rho-1)/2}(\ln^{(v-1)/2}N)\,N^{-1/q}\sqrt{\ln(1/\delta)}. \end{aligned} \tag{33}$$

Substituting (30), (34) and (33) in (25), we have

$$\lambda_{N,\delta}\left(W_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right) \ll (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N) \times\left(1+N^{-1/q}\sqrt{\ln(1/\delta)}\right), \tag{34}$$

which completes the upper estimate of Theorem 1.

Now we proceed to estimate the lower bound. We begin to prove the left inequality of the part (b) of Theorem 1. Let $2\leqslant q<\infty$. It follows from Theorem 4, Lemma 1 and note that $|S|\asymp k^{\nu-1}$, we have

$$\begin{aligned}
&\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)\\
&\gg 2^{-(r_1+1/q+(\rho-1)/2)k}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},\ \upsilon,\ \ell_q^{\|S\|}\right)\\
&\gg 2^{-(r_1+1/q+(\rho-1)/2)k}(\|S\|^{1/q}+\sqrt{\ln(1/\delta)})\\
&\gg 2^{-(r_1+(\rho-1)/2+1/q)k}|S|^{1/q}2^{k/q}+2^{-(r_1+(\rho-1)/2+1/q)k}\sqrt{\ln(1/\delta)}\\
&\gg 2^{-(r_1+(\rho-1)/2)k}k^{(\nu-1)/q}+2^{-(r_1+(\rho-1)/2)k}k^{(\nu-1)/q}k^{-(\nu-1)/q}2^{-k/q}\sqrt{\ln(1/\delta)}\\
&\asymp (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}\ \ln^{(\nu-1)/q}N+(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}\\
&\quad\times(\ln^{(\nu-1)/q}N)N^{-1/q}\sqrt{(\ln 1/\delta)}\\
&\asymp (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/q}N)(1+N^{-1/q}\sqrt{\ln(1/\delta)}).
\end{aligned}$$

We turn to establish the lower estimate for the case $1<q\leqslant 2$. In this case, the lower bound of Theorem 1 can be obtained directly from our paper [2], but for convenience to the reader, we give the proof in details. By Theorem 4, and Lemma 2, we have

$$\begin{aligned}
&\lambda_{N,\delta}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)\\
&\gg 2^{-(r_1+1/q+(\rho-1)/2)k}|S|^{1/2-1/q}\lambda_{N,\delta}\left(\mathbb{R}^{\|S\|},\ \upsilon,\ \ell_q^{\|S\|}\right)\\
&\gg 2^{-(r_1+1/q+(\rho-1)/2)k}|S|^{1/2-1/q}\|S\|^{1/q-1/2}\sqrt{\|S\|+\ln(1/\delta)}\\
&\gg 2^{-(r_1+(\rho-1)/2)k}|S|^{1/2}+2^{-(r_1+(\rho-1)/2+1/2)k}\sqrt{\ln(1/\delta)}\\
&\gg 2^{-(r_1+(\rho-1)/2)k}k^{(\nu-1)/2}+2^{-(r_1+(\rho-1)/2)k}\frac{k^{(\nu-1)/2}}{2^{k/2}k^{(\nu-1)/2}}\sqrt{\ln(1/\delta)}\\
&\asymp (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}\ \ln^{(\nu-1)/2}N\\
&\quad+(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)\frac{1}{\sqrt{N}}\sqrt{(\ln 1/\delta)}\\
&\asymp (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)\sqrt{1+1/N\ln(1/\delta)},
\end{aligned}$$

which is the required lower estimate of part (a) of Theorem 1. The proof of Theorem 1 is completed. □

Proof of Theorem 2. First, we estimate the upper bounds. In this case, we only need to consider the case of $2\leqslant q<\infty$. It follows from the proof of Theorem 1 that $L_q(\mathbb{T}^d)$ has a linear operator T with $\dim T\leqslant N$ such that for any $\delta\in(0,1/2]$ and some subset $G_\delta\subset$

$MW_2^r(\mathbb{T}^d)$ with $\mu(G_\delta)\leqslant\delta$,

$$\lambda\left(MW_2^r(\mathbb{T}^d)\backslash G_\delta,\ T,\ L_q(\mathbb{T}^d)\right) \ll (N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)\times\left(1+N^{-1/q}\sqrt{\ln(1/\delta)}\right). \tag{35}$$

Consider the sequence $\{G_{2^{-k}}\}_{k=0}^{\infty}$ of sets, where $G_1 = MW_2^r(\mathbb{T}^d)$ for $k=0$. Then it follows from estimate (35) that

$$\begin{aligned}
&\int_{MW_2^r(\mathbb{T}^d)} \lambda\left(x, T, L_q(\mathbb{T}^d)\right)^p \mu(dx)\\
&= \sum_{k=0}^{\infty}\int_{G_{2^{-k}}\backslash G_{2^{-k-1}}} \lambda(x, T, L_q(\mathbb{T}^d))^p \mu(dx)\\
&\ll \sum_{k=0}^{\infty}\lambda\left(MW_2^r(\mathbb{T}^d)\backslash G_{2^{-k-1}}, T, L_q(\mathbb{T}^d)\right)^p \mu(G_{2^{-k}})\\
&\ll \sum_{k=0}^{\infty}\left[(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}\ \ln^{(\nu-1)/2}N\right]^p\left(1+(k+1)^{1/2}N^{-1/q}\right)^p 2^{-k}\\
&\ll \left[(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}\ \ln^{(\nu-1)/2}N\right]^p,
\end{aligned}$$

which completes the upper estimate of $\lambda_N^{(a)}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)_p$.

Now we proceed to the lower estimate of Theorem 2, in this case, it is enough to study the case $1<q\leqslant 2$. By virtue of Theorem 1, there exists a constant c such that

$$\begin{aligned}
&\lambda_{N,1/2}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)\\
&> c\left(N^{-1}\ln^{\nu-1}N\right)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)\left(1+(1/N)\ln^{1/2}2\right).
\end{aligned}\tag{36}$$

Next we prove that for any linear operator T of $MW_2^r(\mathbb{T}^d)$ with $\dim T\leqslant N$, there exists a subset $G\subset MW_2^r(\mathbb{T}^d)$ with measure $\mu(G)\geqslant 1/2$ such that

$$\begin{aligned}
&\|x-Tx\|_{L_q^d} > c(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)(1+(1/N)\ln^{1/2}2),\\
&\forall\, x\in G.
\end{aligned}\tag{37}$$

In fact, let

$$\begin{aligned}
G' = \{&x\in MW_2^r(\mathbb{T}^d):\\
&\|x-Tx\|_{L_q^d} > c(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)(1+(1/N)\ln^{1/2}2),\\
&\forall\, x\in\ G\}.
\end{aligned}$$

Then $\mu(G')\geqslant 1/2$. Otherwise, if $\mu(G')<1/2$, then by the definition of linear (N,δ)-width, we have

$$\begin{aligned}
&\lambda_{N,1/2}\left(MW_2^r(\mathbb{T}^d),\ \mu,\ L_q(\mathbb{T}^d)\right)\\
&\leqslant \sup_{x\in MW_2^r(\mathbb{T}^d)\backslash G'}\|x-Tx\|_{L_q^d}\\
&\leqslant c(N^{-1}\ln^{\nu-1}N)^{r_1+(\rho-1)/2}(\ln^{(\nu-1)/2}N)(1+(1/N)\ln^{1/2}2).
\end{aligned}\tag{38}$$

Compare (36) with (38), we have obtained a contradiction. Hence $\mu(G') \geqslant 1/2$. Let $G = G'$. Then G satisfies (37), which implies,

$$\begin{aligned}
&\int_{MW_2^r(\mathbb{T}^d)} \left(\|x - Tx\|_{L_q^d}\right)^p \mu(dx) \\
&\gg \int_G \left(\|x - Tx\|_{L_q^d}\right)^p \mu(dx) \\
&\gg \left[(N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2} \ln^{(\nu-1)/2} N\right]^p (1 + (1/N)\ln^{1/2} 2)^p \mu(G) \\
&\gg \left[(N^{-1}\ln^{\nu-1} N)^{r_1+(\rho-1)/2} \ln^{(\nu-1)/2} N\right]^p .
\end{aligned}$$

which is the required lower estimate of Theorem 2. Theorem 2 is proved. □

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